Optimal Design of Retirement Plans

Workshop ifa & IVW | Manuel Rach | Feb 20, 2020

Motivation

- “Jede zweite Person in Deutschland ist heute älter als 45 und jede fünfte Person älter als 65 Jahre.”[1]
- “Die Niedrigzinsphase ist eigentlich gar keine Phase mehr, sondern zu einem Dauerzustand geworden.”[2]
- Ken Fisher (financial advisor): “I hate annuities... And so should you!”[3]

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[1] Statistisches Bundesamt (2020). Demografischer Wandel. Available online at `https://www.destatis.de/DE/Themen/Querschnitt/Demografischer-Wandel/_inhalt.html`. Accessed on Feb 16, 2020.

[2] Volksbank Freiburg eG (2020). Die Niedrigzinsphase – Ursachen und Hintergründe. Available online at `https://www.volksbank-freiburg.de/magazin/private-finanzen/die-niedrigzinsphase---ursachen-und-hintergruende.html`. Accessed on Feb 16, 2020.

[3] InsuranceNewsNet Magazine (2014). Why Ken Fisher loves to hate annuities. Available at `http://insurancenewsnetmagazine.com/article/why-ken-fisher-loves-to-hate-annuities-2817`. Accessed on Feb 16, 2020.

Consequences

- Pension reform in the second pillar: New pension schemes, tendency towards reduced guarantees
- Innovative products in the third pillar: Revival of tontines (Milevsky and Salisbury (2015), Chen et al. (2019))
- Idea of tontines: Insurer serves only as administrator and shifts the mortality risk towards the policyholders

Objectives and Research Questions

Private retirement plans: Design and risk analysis

1. How can we design new retirement products which lead to a better risk sharing between policyholders and insurers than traditional annuities and tontines?
2. If we allow to combine tontines and annuities, which combination performs best from a policyholder's point of view?
3. How do subjective mortality beliefs affect the perceived relative attractiveness of tontines and annuities, given actuarially fair pricing?

Research papers

1. Chen, A. and Rach, M. (2019). Options on tontines: An innovative way of combining annuities and tontines. *Insurance: Mathematics and Economics*, 89:182–192.

2. Chen, A., Rach, M., and Sehner, T. (2020). On the optimal combination of annuities and tontines. *ASTIN Bulletin: The Journal of the IAA*, 50(1):95–129.

3. Chen, A., Hieber P., and Rach, M. (2019). Optimal retirement products under subjective mortality beliefs. Submitted to special issue of *Insurance: Mathematics and Economics* on behavioral insurance (under review).

Context of existing literature

- Frequently, retirement products are evaluated using expected utility (e.g. Yaari (1965), Milevsky and Huang (2011) and Milevsky and Salisbury (2015))
- Milevsky and Salisbury (2015): Under actuarially fair pricing, annuities yield a higher expected lifetime utility than tontines
- Question: Why can a tontine still be better than an annuity?
  - Milevsky and Salisbury (2015) and Chen et al. (2019) show that tontines can be more beneficial than annuities under risk loadings.
  - Wu et al. (2015) find that subjective mortality beliefs affect the perceived attractiveness of annuities.

Annuity and Tontine

- Following Yaari (1965), we consider continuous-time payment streams
- T is the residual lifetime of the considered individual
- Annuity: $b_A(t) = \mathbb{1}_{\{T>t\}} c(t)$
- Tontine: $b_{OT}(t) = \mathbb{1}_{\{T>t\}} \frac{n}{N(t)} d(t)$
  - n is the number of initial homogeneous policyholders
  - $N(t)$ is the number still alive at time t

Example

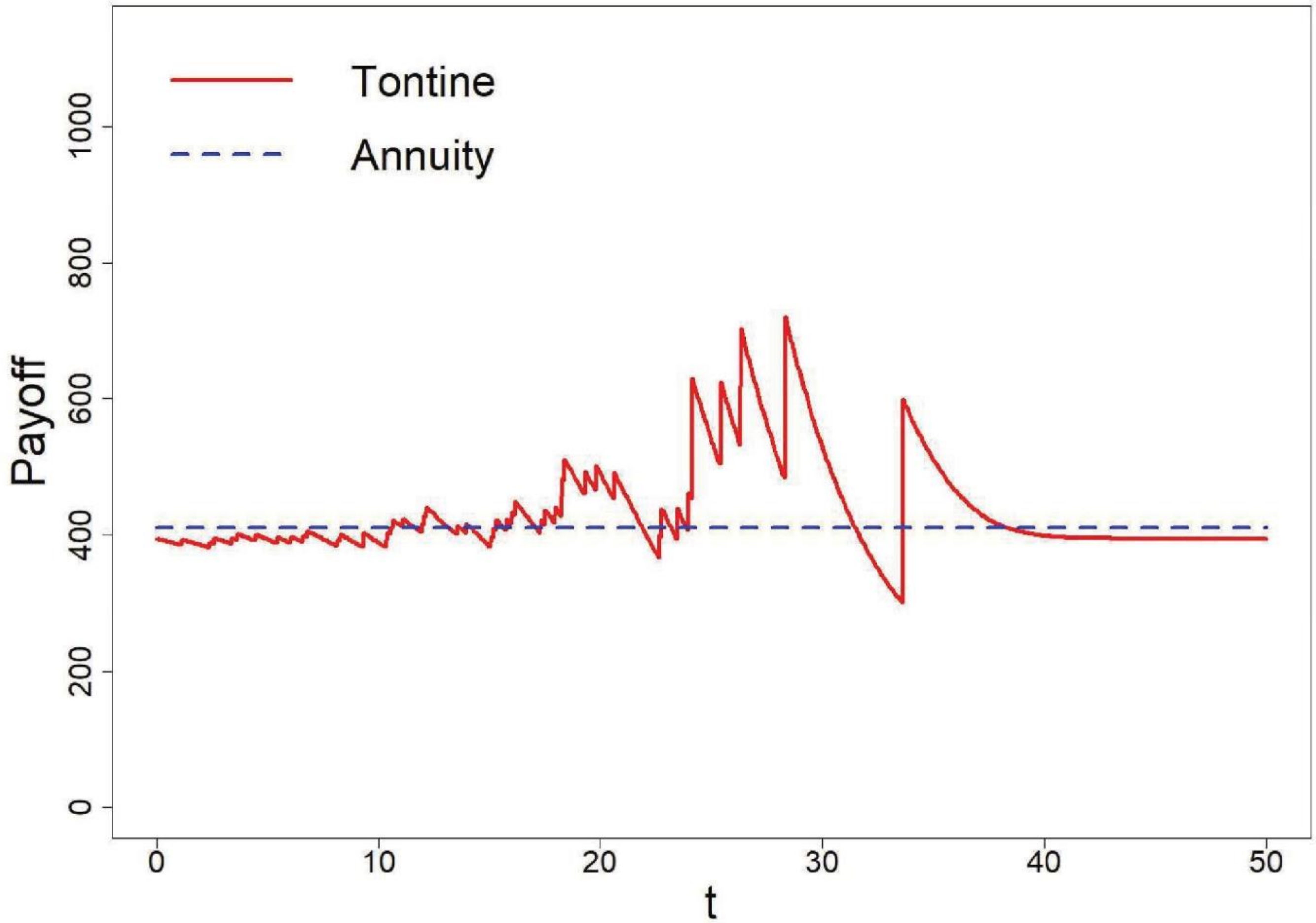

Figure: One path of the payoffs of an annuity and a tontine with identical initial prices of 10000. The pool size is $n = 50$. The plot is based on the assumption that the considered policyholder is alive.

Mortality risk

Unsystematic mortality risk

- Stems from the fact that the lifetime of a person is unknown but still follows some certain mortality law.
- Can initially be diversified by a large pool size

Systematic mortality risk

- Stems from the fact that the true mortality law cannot be determined explicitly.
- Cannot be diversified as it affects the pool as a whole

Mortality

- ${}_tp_x$ is t-year survival probability of x-year old
- Apply random longevity shock ϵ with values in $(-\infty, 1)$ to obtain ${}_tp_x^{1-\epsilon}$
- f_ϵ is the density and m_ϵ is the moment generating function of ϵ.
- T_ϵ is the future lifetime of the individual considered.
- $N_\epsilon(t)$ is the number of policyholders alive at time t.

Paper 1

- Chen, A. and Rach, M. (2019). Options on tontines: An innovative way of combining annuities and tontines. *Insurance: Mathematics and Economics*, 89:182–192.

- Include risk capital charges in the premium and propose a tontine with minimum guarantee which can lead to a better risk sharing between insurers and policyholders

Paper 2

- Chen, A., Rach, M., and Sehner, T. (2019). On the optimal combination of annuities and tontines. *ASTIN Bulletin: The Journal of the IAA*, 50(1):95–129.
- Include risk capital charges in the premium and compare the novel products tonuity (Chen et al. (2019)) and antine to a portfolio consisting of an annuity and a tontine.

Combinations of annuities and tontines

- Payoff of the **portfolio** is given by $b_{AT}(t) = b_A(t) + b_{OT}(t)$.
- Payoff of the **antine** for any switching time $\sigma \in [0, \infty]$:

$$b_{[\sigma]}(t) = \mathbb{1}_{\{0 \leq t < \min\{\sigma, T_\epsilon\}\}} c_{[\sigma]}(t) + \mathbb{1}_{\{\sigma \leq t < T_\epsilon\}} \frac{n}{N_\epsilon(t)} d_{[\sigma]}(t)$$

  Special cases: $\sigma = 0 \Rightarrow$ Tontine, $\sigma = \infty \Rightarrow$ Annuity

- **Tonuity**: Starts as tontine, then switches to annuity (Chen et al. (2019)).

Budget constraint / expected value principle

- Premium of the annuity ($C_A > 0$):

$$P_0^A = \mathbb{E}\left[\int_0^\infty e^{-rt} b_A(t) dt\right] = \int_0^\infty e^{-rt} {}_tp_x m_\epsilon(-\log {}_tp_x)\, c(t)\, dt$$
$$\widetilde{P}_0^A = (1 + C_A) P_0^A$$

- Premium of the tontine ($C_{OT} < C_A$):

$$P_0^{OT} = \int_0^\infty e^{-rt} \int_{-\infty}^1 \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\, d\varphi\, d(t)\, dt$$
$$\widetilde{P}_0^{OT} = (1 + C_{OT}) P_0^{OT}$$

- Premium of the portfolio:

$$\widetilde{P}_0^{AT} = \widetilde{P}_0^{A,AT} + \widetilde{P}_0^{OT,AT}$$

Budget constraint / expected value principle

- Premium of the antine:

$$
\begin{aligned}
P_0^{[\sigma]} &= \mathbb{E}\left[\int_0^\infty e^{-rt} b_{[\sigma]}(t)dt\right] \\
&= \int_0^\sigma e^{-rt}\,{}_tp_x m_\epsilon(-\log {}_tp_x) c_{[\sigma]}(t)dt \\
&\quad + \int_\sigma^\infty e^{-rt} \int_{-\infty}^1 \left(1-\left(1-{}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)d\varphi d_{[\sigma]}(t)dt \\
&=: P_0^{A,\sigma} + P_0^{OT,\sigma}
\end{aligned}
$$

$$
\widetilde{P}_0^{[\sigma]} = (1+C_A)P_0^{A,\sigma} + (1+C_{OT})P_0^{OT,\sigma}
$$

Antine optimization problem

- The policyholder solves

$$\max_{\left(c_{[\sigma]}(t),\, d_{[\sigma]}(t)\right)_{t\in[0,\infty)}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\mathbb{1}_{\{0\le t<\min\{\sigma,T_\epsilon\}\}}\, u\left(c_{[\sigma]}(t)\right)\right.\right.$$
$$\left.\left.+\mathbb{1}_{\{\sigma\le t<T_\epsilon\}}\, u\left(\frac{n}{N_\epsilon(t)}d_{[\sigma]}(t)\right)\right) dt\right]$$
$$\text{subject to}\quad v=\widetilde{P}_0^{[\sigma]}=(1+C_A)\,P_0^{A,\sigma}+(1+C_{OT})\,P_0^{OT,\sigma}.$$

- Admits an explicit solution!

- Tonuity works similar to antine

Portfolio optimization problem

- The policyholder solves

$$\max_{(c_{AT}(t),\, d_{AT}(t))_{t\in[0,\infty)}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\mathbb{1}_{\{T_\epsilon>t\}}u\left(c_{AT}(t)+\frac{n}{N_\epsilon(t)}d_{AT}(t)\right)\,dt\right]$$

$$\text{subject to}\quad v=\widetilde{P}_0^{AT}=\widetilde{P}_0^{A,AT}+\widetilde{P}_0^{OT,AT}.$$

- Does not admit an explicit solution!

Theorem 1

1. *The optimal solution is a 100% investment in the annuity, that is, the solution is* $d_{AT}(t) = 0$, $c_{AT}(t) = c^*_{[0]}(t)$ *and* $\lambda_{AT} = \lambda_{[0]}$, *if and only if* $C_A \leq C_{OT}$.

2. *If and only if* $C_A \geq C_A^{\mathrm{crit}}$, *where*

$$C_A^{\mathrm{crit}} := (1 + C_{OT}) \max_{t \geq 0} \frac{\kappa_{n,\gamma+1,\epsilon}({}_tp_x) \int_{-\infty}^{1} \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\, d\varphi}{\kappa_{n,\gamma,\epsilon}({}_tp_x)\, {}_tp_x\, m_\epsilon(-\ln {}_tp_x)} - 1,$$

   the optimal solution is a 100% investment in the tontine, that is, the solution is $d_{AT}(t) = d^*_{[\infty]}(t)$, $c_{AT}(t) = 0$ *and* $\lambda_{AT} = \lambda_{[\infty]}$.

3. *Consequently, if and only if*

$$C_{OT} < C_A < C_A^{\mathrm{crit}},$$

   the optimal solution is investing in both annuity and tontine.

Theorem 2

We denote by U_{AT}, $U_{[\tau]}$ and $U_{[\sigma]}$ the optimal levels of expected utility resulting from the optimal portfolio, tonuity and antine, respectively. Then it holds

$$U_{AT} \geq U_{[\tau]}, \quad U_{AT} \geq U_{[\sigma]}$$

for all switching times τ, σ and risk loadings C_A and C_{OT}.

Paper 3

- Chen, A., Hieber P., and Rach, M. (2019). Optimal retirement products under subjective mortality beliefs. Submitted to special issue of *Insurance: Mathematics and Economics* on behavioral insurance (under review).

- Remain in the actuarially fair pricing framework and compare annuities to tontines under subjective mortality beliefs.

Subjective mortality beliefs

- ${}_tp_x$ is the survival probability used by the insurer
- ${}_t\tilde{p}_x$ used by the individual for herself
- ${}_t\hat{p}_x$ used by the individual for everyone else
- Apply random longevity shock ϵ with values in $(-\infty, 1)$ to all three survival curves
- All the payoffs and the expected utility are determined based on the policyholder's subjective beliefs.

Optimization problems

- Annuity:

$$\max_{c(t)} \int_0^\infty e^{-\rho t}\,{}_t\tilde{p}_x \cdot m_\epsilon(-\ln {}_t\tilde{p}_x)\, u(c(t))\,\mathrm{d}t$$

$$\text{s.t.} \quad v = P_0^A = \int_0^\infty e^{-rt}\,{}_tp_x \cdot m_\epsilon(-\ln {}_tp_x)\, c(t)\,\mathrm{d}t$$

- Tontine:

$$\max_{d(t)} \int_0^\infty e^{-\rho t}\, u(d(t))\, \kappa_{n,\gamma,\epsilon}({}_t\hat{p}_x, {}_t\tilde{p}_x)\,\mathrm{d}t$$

$$\text{s.t.} \quad v = P_0^T = \int_0^\infty e^{-rt} \int_{-\infty}^1 \left(1 - \left(1 - {}_tp_x^{1-\varphi}\right)^n\right) f_\epsilon(\varphi)\,\mathrm{d}\varphi\, d(t)\,\mathrm{d}t$$

Theorem 3

(a) If beliefs do not differ between policyholder and insurance company, that is if ${}_t p_x = {}_t\tilde{p}_x = {}_t\hat{p}_x$, we find that the CE of a tontine **never** (that is for any portfolio size $n \in \mathbb{N}$) **exceeds** the CE of an annuity.

(b) If

$$\int_{-\infty}^{1} {}_t p_x^{1-\varphi} f_\epsilon(\varphi)\mathrm{d}\varphi > \left(\frac{\int_{-\infty}^{1} {}_t\tilde{p}_x^{1-\varphi} \left(\frac{1}{{}_t\hat{p}_x^{1-\varphi}} \right)^{1-\gamma} f_\epsilon(\varphi)\mathrm{d}\varphi}{\int_{-\infty}^{1} {}_t\tilde{p}_x^{1-\varphi} f_\epsilon(\varphi)\mathrm{d}\varphi} \right)^{\frac{1}{\gamma-1}}, \qquad (1)$$

there exists a pool size $N_0 \in \mathbb{N}$ such that the subjective CE of a tontine is (for any portfolio size $n \geq N_0$) **higher** than the subjective CE of an annuity.

(c) Consider the case **without systematic mortality risk** ($\epsilon \equiv 0$). In this case, assumption (1) simplifies to ${}_t\hat{p}_x < {}_t p_x$.

Conclusion

- Tontines can be preferred to annuities under safety loadings and subjective mortality beliefs.
- The optimal portfolio of an annuity and a tontine outperforms any antine and tonuity.
- If a policyholder underestimates the lifetimes of the other policyholders relative to the insurer, a tontine with a large enough pool size is superior to an annuity.

References

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